

$\mathcal{N} = 1$ super Yang-Mills from D branes ¹

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Abstract

We use fractional and wrapped branes to describe perturbative and non-perturbative properties of $\mathcal{N} = 1$ super Yang-Mills living on their world-volume.

1 Introduction

String theory emerged in the beginning of the seventies from the attempt to find a theory describing strong interactions [1]. It was soon realized that it contained many unphysical features for describing strong interactions as for instance the existence of massless spin 1 and 2 particles in the hadronic spectrum and extra dimensions. For these reasons it was later proposed [2] that string theories were more suited to describe a unifying theory of all interactions and a description of strong interactions based on some kind of string theory is still to be found. In the meantime it became clear that strong interactions are described by QCD that is a nonabelian gauge theory based on the gauge group $SU(3)$.

The discovery of D branes as non-perturbative states of string theory has open again the way to use string theory for describing the properties of the gauge theories that live on their world-volume. This has been possible because the D branes have the twofold property of being, on the one hand, a classical solution of the low-energy string effective action containing closed string states and of containing, on the other hand, a gauge theory living on its world-volume whose degrees of freedom correspond to open strings having their end-points attached to the world-volume of the D brane. This is a direct consequence of open/closed string duality according to which open-string loop diagrams can equivalently be described by tree diagrams of closed strings. The most impressive consequence of this duality has been

¹Work partially supported by the European Commission RTN programme HPRN-CT-2000-00131.

the Maldacena conjecture [3] that implies the exact equivalence between four-dimensional $\mathcal{N} = 4$ super Yang-Mills and type IIB string theory compactified on $AdS_5 \times S^5$.

Recently these ideas have been extended to more realistic gauge theories that are less supersymmetric and non-conformal and although in this case no exact duality has been established, nevertheless they have allowed to derive perturbative and non-perturbative properties of the gauge theories living on the world-volume of D branes from their supergravity description ¹ Those more realistic gauge theories can be obtained by considering more sophisticated D branes as fractional D branes of some orbifold that are stack at the orbifold fixed point or as branes wrapped on some nontrivial two-cycle of a Calabi-Yau space.

In this talk I want to discuss the results obtained restricting myself to $\mathcal{N} = 1$ super Yang-Mills where also non-perturbative properties as the gaugino condensantion and the non-perturbative effective potentials have been derived. It turns out that, in order to derive non-perturbative properties, we need to consider supergravity solutions that are regular also at short distances. Two of them are known, namely the one corresponding to a D5 brane wrapped on a nontrivial two-cycle of a Calabi-Yau space [9] and the one corresponding to the deformed conifold found in Ref. [10]. In the next section we will describe them and from them we will derive the properties of $\mathcal{N} = 1$ super Yang-Mills.

An important property of the gauge theories that can be derived with these methods are the running coupling constant that is given by the following general formula:

$$\frac{4\pi}{g_{YM}^2} = \frac{1}{g_s(2\pi\sqrt{\alpha'})^2} \int d^2\xi e^{-(\phi-\phi_0)} \sqrt{\det(G_{AB} + B_{AB})} \quad (1)$$

and the θ parameter given by:

$$\theta_{YM} = \frac{1}{2\pi\alpha'g_s} \int_{\mathcal{C}_2} (C_2 + C_0 B_2) \quad (2)$$

The previous formulas are valid for both fractional and wrapped branes as it can be found in Ref.s [5, 7, 8].

2 $\mathcal{N} = 1$ super Yang-Mills from D branes

In this section we will review the properties of the two regular solutions of ten-dimensional type IIB string theory discussed in the introduction and we will extract from them the properties of $\mathcal{N} = 1$ super Yang-Mills.

¹For extensive reviews of these developments see Refs. [4, 5, 6, 7, 8].

2.1 The MN solution

We start by writing the classical solution corresponding to N D branes wrapped on a two-cycle of a non-compact Calabi-Yau space found in Ref. [9]. It has a non-trivial metric:

$$ds_{10}^2 = e^\Phi \left[dx_{1,3}^2 + \frac{e^{2h}}{\lambda^2} (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2) \right] + \frac{e^\Phi}{\lambda^2} \left[d\rho^2 + \sum_{a=1}^3 (\sigma^a - \lambda A^a)^2 \right] , \quad (3)$$

a two-form R-R potential

$$C^{(2)} = \frac{1}{4\lambda^2} \left[(\psi + \psi_0) (\sin \theta' d\theta' \wedge d\phi - \sin \tilde{\theta} d\tilde{\theta} \wedge d\tilde{\varphi}) - \cos \theta' \cos \tilde{\theta} d\phi \wedge d\tilde{\varphi} \right] + \frac{a}{2\lambda^2} [d\tilde{\theta} \wedge \sigma^1 - \sin \tilde{\theta} d\tilde{\varphi} \wedge \sigma^2] \quad (4)$$

and a dilaton

$$e^{2\Phi} = \frac{\sinh 2\rho}{2e^h} , \quad (5)$$

where

$$e^{2h} = \rho \coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4} , \quad e^{2k} = e^h \frac{\sinh 2\rho}{2} , \quad a = \frac{2\rho}{\sinh 2\rho} \quad (6)$$

and

$$A^1 = -\frac{1}{2\lambda} a(r) d\tilde{\theta} , \quad A^2 = \frac{1}{2\lambda} a(r) \sin \tilde{\theta} d\tilde{\varphi} , \quad A^3 = -\frac{1}{2\lambda} \cos \tilde{\theta} d\tilde{\varphi} . \quad (7)$$

with $\rho \equiv \lambda r$ and $\lambda^{-2} = Ng_s \alpha'$. The left-invariant 1-forms of S^3 are

$$\sigma^1 = \frac{1}{2} [\cos \psi d\theta' + \sin \theta' \sin \psi d\phi] , \quad \sigma^2 = -\frac{1}{2} [\sin \psi d\theta' - \sin \theta' \cos \psi d\phi] , \quad \sigma^3 = \frac{1}{2} [d\psi + \cos \theta' d\phi] , \quad (8)$$

with $0 \leq \theta' \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$. The variables $\tilde{\theta}$ and $\tilde{\varphi}$ describe a two-dimensional sphere and vary in the range $0 \leq \tilde{\theta} \leq \pi$ and $0 \leq \tilde{\varphi} \leq 2\pi$.

We can now use the previous solution for computing the running coupling constant and the θ parameter of $\mathcal{N} = 1$ super Yang-Mills ². In order to do that we have to fix the cycle around which we should perform the integrals in eq.s (1) and (2). It turns out that this two-cycle is specified by:

$$\tilde{\theta} = \theta' , \quad \tilde{\varphi} = -\phi , \quad \psi = 0 \quad (9)$$

²See Ref.s [5, 7, 8] and Ref.s therein.

keeping ρ fixed. If we now compute the gauge couplings on the cycle specified in the previous equation ($B_2 = C_0 = 0$) we get

$$\frac{4\pi^2}{Ng_{YM}^2} = \rho \coth 2\rho + \frac{1}{2}a(\rho) \cos \psi \quad (10)$$

and

$$\theta_{YM} = \frac{1}{2\pi g_s \alpha'} \int_{S^2} C_2 = -N (\psi + a(\rho) \sin \psi + \psi_0) \quad (11)$$

where we have not yet taken $\psi = 0$ for reasons that will become clear in a moment. Eq. (10) shows that the coupling constant is running as a function of the distance ρ from the branes. In order to obtain the correct running of the gauge theory we have to find a relation between ρ and the renormalization group scale μ . This can be obtained with the following considerations. If we look at the Maldacena-Núñez solution it is easy to see that the metric in Eq.(3) is invariant under the following transformations:

$$\begin{cases} \psi \rightarrow \psi + 2\pi & \text{if } a \neq 0 \\ \psi \rightarrow \psi + 2\epsilon & \text{if } a = 0 \end{cases} \quad (12)$$

where ϵ is an arbitrary constant. On the other hand C_2 is not invariant under the previous transformations but its flux, that is exactly equal to θ_{YM} in Eq.(11), changes by an integer multiple of 2π . In fact one gets:

$$\theta_{YM} = \frac{1}{2\pi \alpha' g_s} \int_{C_2} C_2 \rightarrow \theta_{YM} + \begin{cases} -2\pi N & , \text{ if } a \neq 0 \\ -2N\epsilon & , \text{ if } a = 0, \epsilon = \frac{\pi k}{N} \end{cases} \quad (13)$$

This changes θ_{YM} by a factor 2π times an integer. But since the physics does not change when $\theta_{YM} \rightarrow \theta_{YM} + 2\pi$ this implies that the transformation in Eq.(13) is an invariance. Notice that also Eq.(10) for the gauge coupling constant, is invariant under the transformation in Eq.(12). This means that the classical solution and also the gauge couplings are invariant under the Z_2 transformation if $a \neq 0$, while this symmetry becomes Z_{2N} if a is taken to be zero. This implies that, since in the ultraviolet $a(\rho)$ is exponentially small, we can neglect it and we have a Z_{2N} symmetry, while in the infrared we cannot neglect $a(\rho)$ anymore and we have only a Z_2 symmetry left. It is on the other hand well known that $\mathcal{N} = 1$ super Yang-Mills has a non zero gaugino condensate $\langle \lambda\lambda \rangle$ that is responsible for the breaking of Z_{2N} into Z_2 . Therefore it is natural to identify the gaugino condensate with the function $a(\rho)$ that appears in the supergravity solution:

$$\langle \lambda\lambda \rangle \sim \Lambda^3 = \mu^3 a(\rho) \quad (14)$$

This gives the relation between the renormalization group scale μ and the supergravity space-time parameter ρ . In the ultraviolet (large ρ) $a(\rho)$ is exponentially suppressed and in Eq.s (10) and (11) we can neglect it obtaining:

$$\frac{4\pi^2}{Ng_{YM}^2} = \rho \coth 2\rho \quad , \quad \theta_{YM} = -N(\psi + \psi_0) \quad (15)$$

The chiral anomaly can be obtained by performing the transformation $\psi \rightarrow \psi + 2\epsilon$ and getting:

$$\theta_{YM} \rightarrow \theta_{YM} - 2N\epsilon \quad (16)$$

This implies that the Z_{2N} transformations corresponding to $\epsilon = \frac{\pi k}{N}$ are symmetries because they shift θ_{YM} by multiples of 2π .

In general, however, Eq.s (10) and (11) are only invariant under the Z_2 subgroup of Z_{2N} corresponding to the transformation:

$$\psi \rightarrow \psi + 2\pi \quad (17)$$

that changes θ_{YM} in Eq.(11) as follows

$$\theta_{YM} \rightarrow \theta_{YM} - 2N\pi \quad (18)$$

leaving invariant the gaugino condensate:

$$\langle \lambda^2 \rangle = \frac{\mu^3}{3Ng_{YM}^2} e^{-\frac{8\pi^2}{Ng_{YM}^2}} e^{i\theta_{YM}/N} \quad (19)$$

Therefore the chiral anomaly and the breaking of Z_{2N} to Z_2 are encoded in Eq.s (10) and (11). Finally, if we put $\psi = 0$ in eq. (10) and we consider it together with eq.(14) we can determine the running coupling constant as a function of μ :

$$\frac{4\pi^2}{Ng_{YM}^2} = \rho \coth 2\rho - \frac{1}{2}a(\rho) = \rho \tanh \rho \quad (20)$$

This equation taken together with eq.(14) reproduces [7] the NSVZ β -function plus non-perturbative corrections due to fractional instantons.

2.2 The conifold solution

In this second subsection we will start presenting the classical solutions of type IIB supergravity corresponding to have N fractional D branes located at the tip of the conifold and M bulk D branes respectively for the singular and the deformed conifold. Then we will use them to get information on $\mathcal{N} = 1$ super Yang-Mills.

The conifold is a manifold described by the following equation between complex variables:

$$x^2 + y^2 + z^2 + t^2 = 0 \quad (21)$$

that can be seen as the six-dimensional cone over the space $T^{1,1}$, so that the metric can be written as

$$ds_6^2 = dr^2 + r^2 ds_{T^{1,1}}^2 \quad (22)$$

where

$$ds_{T^{1,1}}^2 = \frac{1}{9}(g^5)^2 + \frac{1}{6} \sum_{i=1}^4 (g^i)^2 \quad (23)$$

and

$$g^1 = \frac{e^1 - e^3}{\sqrt{2}} , \quad g^2 = \frac{e^2 - e^4}{\sqrt{2}} , \quad g^3 = \frac{e^1 + e^3}{\sqrt{2}} , \quad g^4 = \frac{e^2 + e^4}{\sqrt{2}} , \quad g^5 = e^5 \quad (24)$$

with

$$\begin{aligned} e^1 &= -\sin \theta_1 d\phi_1 , \quad e^2 = d\theta_1 , \quad e^3 = \cos \psi \sin \theta_2 d\phi_2 - \sin \psi d\theta_2 \\ e^4 &= \sin \psi \sin \theta_2 d\phi_2 + \cos \psi d\theta_2 , \quad e^5 = d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2 \end{aligned} \quad (25)$$

The range of the angular coordinates is defined to be:

$$0 \leq \psi \leq 4\pi , \quad 0 \leq \theta_1, \theta_2 \leq \pi , \quad 0 \leq \phi_1, \phi_2 \leq 2\pi . \quad (26)$$

Topologically the $T^{1,1}$ manifold can be thought of as $S^2 \times S^3$. The two cycles are identified by:

$$\begin{aligned} S^2 &: \quad \psi = 0, \quad \theta_1 = \theta_2, \quad \phi_1 = -\phi_2, \\ S^3 &: \quad \theta_1 = \phi_1 = 0 \end{aligned} \quad (27)$$

Their volume forms are given respectively by:

$$\omega_3 = \frac{1}{2} g^5 \wedge (g^1 \wedge g^2 + g^3 \wedge g^4) , \quad \omega_2 = \frac{1}{2} (g^1 \wedge g^2 + g^3 \wedge g^4) \quad (28)$$

normalized as follows:

$$\int_{S^2} \omega_2 = 4\pi , \quad \int_{S^3} \omega_3 = 8\pi^2 \quad (29)$$

They satisfy the following duality relation in the six-dimensional space transverse to the world-volume of a D3 brane:

$$*^6 \left(\omega_2 \wedge \frac{dr}{r} \right) = \frac{\omega_3}{3} , \quad *^6 \omega_3 = -3\omega_2 \wedge \frac{dr}{r} \quad (30)$$

The classical solution corresponding to N fractional at the tip of the conifold and M bulk branes is given by [11]:

$$ds^2 = h^{-1/2}(r)\eta_{\alpha\beta}dx^\alpha dx^\beta + h^{1/2}(r)\left(dr^2 + r^2 ds_{T^{1,1}}^2\right) \quad (31)$$

$$B_2 = \frac{3g_s N \alpha'}{2} \log \frac{r}{r_0} \omega_2 \quad , \quad H_3 \equiv dB_2 = \frac{3g_s N \alpha'}{2} \frac{dr}{r} \wedge \omega_2 \quad (32)$$

$$F_3 = \frac{g_s N \alpha'}{2} \omega_3 \quad (33)$$

$$\tilde{F}_5 = \mathcal{F}_5 + {}^* \mathcal{F}_5 \quad . \quad \mathcal{F}_5 = 27\pi g_s (\alpha')^2 M_{eff}(r) Vol(T^{1,1}) \quad , \quad (34)$$

where r_0 is a regulator,

$$h(r) = 27\pi (\alpha')^2 \frac{g_s M + \frac{3(g_s N)^2}{2\pi} \left(\log \frac{r}{r_0} + \frac{1}{4}\right)}{4r^2} \quad , \quad (35)$$

$$M_{eff}(r) = M + \frac{3g_s N^2}{2\pi} \log \frac{r}{r_0} \quad (36)$$

The previous solution, corresponding to the singular conifold described by eq. (21), has a naked singularity at short distances when $r = r_0$. To remove this singularity the equation that defines the conifold is replaced by:

$$x^2 + y^2 + z^2 + t^2 = \epsilon^2 \quad (37)$$

This corresponds to blow up the three-sphere S^3 of $T^{(1,1)}$. The metric of the deformed conifold is given by [12, 13, 14]:

$$ds_6^2 = \frac{\epsilon^{4/3}}{2} K(\tau) \left[\frac{d\tau^2 + (g^5)^2}{3K^3(\tau)} + \cosh^2 \frac{\tau}{2} \left((g^3)^2 + (g^4)^2 \right) + \sinh^2 \frac{\tau}{2} \left((g^1)^2 + (g^2)^2 \right) \right] \quad (38)$$

where

$$K(\tau) = \frac{(\sinh 2\tau - 2\tau)^{1/3}}{2^{1/3} \sinh \tau} \quad (39)$$

The complete solution is given by [10]:

$$ds^2 = h^{-1/2}(\tau)\eta_{\alpha\beta}dx^\alpha dx^\beta + h^{1/2}(\tau)ds_6^2 \quad (40)$$

$$B_2 = \frac{g_s N \alpha'}{2} \left[f(\tau)(g^1 \wedge g^2) + k(\tau)(g^3 \wedge g^4) \right] \quad (41)$$

$$F_3 = \frac{g_s N \alpha'}{2} \left[g^5 \wedge g^3 \wedge g^4 + d \left(F(\tau)(g^1 \wedge g^3 + g^2 \wedge g^4) \right) \right] \quad (42)$$

$$\tilde{F}_5 = \mathcal{F}_5 + {}^* \mathcal{F}_5 \quad . \quad \mathcal{F}_5 = \frac{g_s N^2 (\alpha')^2}{4} \ell(\tau) g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5 \quad , \quad (43)$$

where

$$F(\tau) = \frac{\sinh \tau - \tau}{2 \sinh \tau} , \quad f(\tau) = \frac{\tau \coth \tau - 1}{2 \sinh \tau} (\cosh \tau - 1) \quad (44)$$

and

$$k(\tau) = \frac{\tau \coth \tau - 1}{2 \sinh \tau} (\cosh \tau + 1) , \quad \ell(\tau) = \frac{\tau \coth \tau - 1}{4 \sinh^2 \tau} (\sinh 2\tau - 2\tau) \quad (45)$$

$$h(\tau) = (g_s N \alpha')^2 2^{2/3} \epsilon^{-8/3} \int_{\tau}^{\infty} dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh 2x - 2x)^{1/3} \quad (46)$$

For large values of τ the previous regular solution behaves as the singular one. In particular by identifying eq. (38) with the six-dimensional part of eq. (31) without the warp factor we get a relation between τ and r given by:

$$\tau = 3 \log \frac{r}{r_0} , \quad r_0 \equiv \frac{3^{1/2} \epsilon^{2/3}}{2^{5/6}} \quad (47)$$

Let us now use the previous solution for obtaining the gauge couplings of $\mathcal{N} = 1$ super Yang-Mills given in eq.s (1) and (2). In the case of the conifold eq.s (1) and (2) become:

$$\frac{4\pi}{g_{YM}^2} = \frac{1}{g_s (2\pi \sqrt{\alpha'})^2} \int_{S^2} B_2 , \quad \theta_{YM} = \frac{1}{2\pi \alpha' g_s} \int_{S^2} C_2 \quad (48)$$

We start by computing the θ_{YM} parameter. We need to extract C_2 from eq. (42). It is given by:

$$\begin{aligned} C_2 = & \frac{N g_s \alpha'}{4} [(\psi + \psi_0) (\sin \theta_2 d\phi_2 \wedge d\theta_2 - \sin \theta_1 d\phi_1 \wedge d\theta_1) + \\ & - d\phi_1 \wedge d\phi_2 \cos \theta_1 \cos \theta_2 + (1 - 2F(\tau)) (\sin \psi (\sin \theta_2 d\phi_2 \wedge d\theta_1 - \sin \theta_1 d\phi_1 \wedge d\theta_2) + \\ & + \cos \psi (d\theta_1 \wedge d\theta_2 - \sin \theta_1 \sin \theta_2 d\phi_1 \wedge d\phi_2))] \end{aligned} \quad (49)$$

When we insert C_2 in the second equation in (48) and we take $\theta_1 = \theta_2, \phi_1 = -\phi_2$ we get

$$\theta_{YM} = N [(\psi + \psi_0) + \sin \psi (1 - 2F(\tau))] \quad (50)$$

showing as in the case of the Maldacena-Núñez solution that, for large values of τ where $F(\tau) = \frac{1}{2}$ the $U(1)_R$ is broken down to Z_{2N} . This is also the symmetry of the asymptotic solution, while the full solution presented above does not have anymore this Z_{2N} symmetry. It is further broken to Z_2 that is the remaining symmetry of $\mathcal{N} = 1$ super Yang-Mills that leaves invariant the gaugino condensate. This observation gives us the possibility of again

identifying the gluino condensate. The natural thing is to identify it with $2F(\tau)$ after having subtracted its asymptotic value [15]:

$$1 - 2F(\tau) = \frac{\tau}{\sinh \tau} \quad (51)$$

This quantity will play the same role as $a(\rho)$ in eq. (6) and allows one to establish the relation between the supergravity parameter τ and the renormalization group scale μ :

$$\frac{\Lambda^3}{\mu^3} = \frac{\tau}{\sinh \tau} \quad (52)$$

Notice the strong similarity between this equation and eq. (14) and also between eq.s (11) and (50). The running coupling constant can then be computed by inserting eq.(41) in the first equation in (48) getting:

$$\frac{4\pi^2}{g_{YM}^2} = Nk(\tau) = N \frac{\tau \coth \tau - 1}{2 \sinh \tau} (\cosh \tau + 1) \quad (53)$$

that taken together with eq.(52) allows one to compute the β -function of $\mathcal{N} = 1$ super Yang-Mills. A simple calculation shows that

$$\beta(g_{YM}) \equiv \mu \frac{\partial g_{YM}}{\partial \mu} = -\frac{3Ng_{YM}^3}{16\pi^2} \cdot \frac{1 + \frac{\tau}{\sinh \tau}}{\coth \tau - \frac{1}{\tau}} \quad (54)$$

where τ is a function of g_{YM} given by eq.(53). In the ultraviolet ($\tau \rightarrow \infty$) one can get a more explicit relation between them given by:

$$\frac{1}{\tau} = \frac{\frac{Ng_{YM}^2}{8\pi^2}}{1 + \frac{Ng_{YM}^2}{8\pi^2}} \cdot \left[1 + \frac{e^{-8\pi^2/(Ng_{YM}^2)}}{1 + \frac{Ng_{YM}^2}{8\pi^2}} \right] \quad (55)$$

Inserting it in eq.(54) and neglecting the contribution of the fractional instantons we get:

$$\beta(g_{YM}) = -\frac{3Ng_{YM}^3}{16\pi^2} \cdot \frac{1}{1 - \frac{Ng_{YM}^2}{8\pi^2} [1 + \frac{Ng_{YM}^2}{8\pi^2}]^{-1}} \quad (56)$$

that agrees with the NSVZ β -function up to two loops, but differs from it for higher loops. Notice that, if one takes into account only the leading asymptotic behaviour of the right hand side of eq.(53), one gets exactly the NSVZ β -function [15].

In the case of the Klebanov-Strassler solution one can also compute the effective potential of $\mathcal{N} = 1$ super Yang-Mills, namely the Veneziano-Yankielowicz potential, following a proposal by Vafa [16] where he identifies

it with the superpotential of $\mathcal{N} = 1$ supergravity that is given by the following expression:

$$W_{eff} \sim \sum_i \left[\int_{A_i} G_3 \int_{B_i} \Omega - \int_{A_i} \Omega \int_{B_i} G_3 \right] \quad (57)$$

in terms of the flux of $G_3 = F_3 + iH_3$ and of the periods of the holomorphic $(3, 0)$ -form Ω of Calabi-Yau threefold under consideration. $A_i(B_i)$ are the compact (noncompact) orthogonal three-cycles of a Calabi-Yau manifold.

In the following we will use the previous general formula in the case of the deformed conifold solution and we will obtain [17] the Veneziano-Yankielowicz potential³. In the case of the conifold we have only one compact and one non-compact cycle specified by

$$A(compact) \quad : \quad r = constant, \theta_1 = 0, \phi_1 = 0 \quad (58)$$

$$B(noncompact) \quad , \quad \psi = 0, \theta_1 = \theta_2, \phi_1 = -\phi_2 \quad (59)$$

Let us start computing the fluxes of G_3 along the two cycles. For both the singular and deformed conifold solution the flux of G_3 on the compact cycle is given by:

$$\frac{1}{(2\pi\sqrt{\alpha'})^2 g_s} \int_A G_3 = N \quad (60)$$

In the case of the non compact cycle in order to get finite expressions we need to introduce a cut-off r_c for large values of r . In addition in the case of the singular solution we also need to introduce a cut-off r_0 for small values of r . In the case of the singular solution we get:

$$\int_B F_3 = 0 \quad , \quad \int_B H_3 = 6\pi g_s N \alpha' \int_{r_0}^{r_c} \frac{dr}{r} = 6\pi g_s N \alpha' \log \frac{r_c}{r_0} \quad (61)$$

while in the case of the deformed we get:

$$\int_B F_3 = 0 \quad , \quad \int_B H_3 = 4\pi g_s N \alpha' \int_0^{\tau_c} d\tau k'(\tau) = 4\pi g_s N \alpha' k(\tau_c) \quad (62)$$

that implies

$$\frac{1}{(2\pi\sqrt{\alpha'})^2 g_s} \int_B G_3 = \frac{2Nk(\tau_c)}{2\pi i} \sim \frac{2N\tau_c}{4\pi i} = \frac{3N}{2\pi i} \log \frac{r_c}{r_0} \quad (63)$$

where we have used eq.(47) and the behaviour of $k(\tau_c) \sim \frac{\tau_c}{2}$ for large τ_c .

³The procedure followed here is taken from Ref. [8].

Let us now compute the two periods of the holomorphic $(3,0)$ -form Ω . The deformed conifold is described by the equation:

$$F = x^2 + y^2 + z^2 + t^2 - \epsilon^2 = 0 \quad (64)$$

and Ω is defined as

$$\Omega = \frac{1}{2\pi i} \oint_{F=0} \frac{dx \wedge dy \wedge dz \wedge dt}{F} = \frac{dx \wedge dy \wedge dz}{2\sqrt{\epsilon^2 - x^2 - y^2 - z^2}} \quad (65)$$

The cycle A is determined by letting x and y to run in the intervals $-\epsilon \leq x \leq \epsilon$; $-\sqrt{\epsilon^2 - x^2} \leq y \leq \sqrt{\epsilon^2 - x^2}$ and z around the branch cut that connects the two branch points $\pm\sqrt{\epsilon^2 - x^2 - y^2}$. We have to compute therefore the following expression:

$$\int_A \Omega = \int_{-\epsilon}^{\epsilon} dx \int_{-\sqrt{\epsilon^2 - x^2}}^{\sqrt{\epsilon^2 - x^2}} dy \int_{\gamma} \frac{dz}{2\sqrt{\epsilon^2 - x^2 - y^2 - z^2}} \quad (66)$$

where γ is a curve around the branch cut. The integral over z can be computed by deforming it to an integral around infinity and one gets:

$$\int_A \Omega = \int_{-\epsilon}^{\epsilon} dx \int_{-\sqrt{\epsilon^2 - x^2}}^{\sqrt{\epsilon^2 - x^2}} dy \oint_{\infty} \frac{dz}{2iz} = \int_{-\epsilon}^{\epsilon} dx 2\pi \sqrt{\epsilon^2 - x^2} = \pi^2 \epsilon^2 \quad (67)$$

In the case of the cycle B we get instead:

$$\begin{aligned} \int_B \Omega &= \int_{\epsilon}^{r_c^{3/2}} dx 2\pi \sqrt{\epsilon^2 - x^2} = 2\pi \epsilon^2 \int_{\pi/2}^{\arcsin \frac{r_c^{3/2}}{\epsilon}} d\alpha \cos^2 \alpha = \\ &= \pi r_c^{3/2} \sqrt{\epsilon^2 - r_c^3} + \pi \epsilon^2 \arcsin \frac{r_c^{3/2}}{\epsilon} - \frac{1}{2} \pi^2 \epsilon^2 \end{aligned} \quad (68)$$

where we have taken care that the complex coordinates in eq.(64) have dimension $L^{3/2}$ as you can see from eq.(47). Expanding eq. (68) for large values of r_c we get:

$$\int_B \Omega = 2\pi i \left[\frac{r_c^3}{2} - \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} \log \frac{\epsilon^2}{4} - \frac{\epsilon^2}{2} \log r_c^{3/2} \right] \quad (69)$$

Putting together eq.s (60), (63), (67) and (69) we get the following effective potential for $\mathcal{N} = 1$ super Yang-Mills:

$$W_{eff} = -\frac{1}{2\pi i} \frac{1}{(2\pi\sqrt{\alpha'})^2 g_s} \frac{1}{(2\pi\alpha')^3} \left[\int_A G_3 \int_B \Omega - \int_A \Omega \int_B G_3 \right] =$$

$$= -\frac{N}{(2\pi\alpha')^3} \left[3\frac{\epsilon^2}{4} \log \frac{r_c}{r_0} + \frac{r_c^3}{2} - \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} \log \frac{\epsilon^2}{4r_c^3} \right] \quad (70)$$

Making the following identifications:

$$r_c = 2\pi\alpha'\mu \quad , \quad r_0 = 2\pi\alpha'\Lambda \quad , \quad \frac{\epsilon^2}{4} = (2\pi\alpha')^3 S \quad (71)$$

and neglecting the constant term we get the Veneziano-Yankielowicz effective superpotential:

$$W_{eff} = NS \left(1 - \log \frac{S}{\Lambda^3} \right) \quad (72)$$

Finally let me mention that the previous procedure for computing the effective superpotential has been also used for computing [18] the Affleck-Dine-Seiberg superpotential in the case of the orbifold $C^3/(Z_2 \times Z_2)$.

Acknowledgements

I thank E. Imeroni for explaining me many details of his PhD thesis and A. Lerda and P. Merlatti for discussions.

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